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ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE

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1. INTRODUCTION

Let X and Y be n - and m - dimensional complex manifolds, respectively.

$$S^*X := (T^*X - X)/\mathbb{R}^+, S^*Y := (T^*Y - Y)/\mathbb{R}^+.$$

We define the mapping γ as

$$\gamma : \overset{\circ\circ}{T}^*(X \times Y) \ni (z, w; \xi, \eta) \longmapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

where

$$\overset{\circ\circ}{T}^*(X \times Y) := T^*(X \times Y) \setminus \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$

For $d > 0$ and an open subset U of $S^*X \times S^*Y$ we denote

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$$

by $\gamma^{-1}(U; d, d)$.

Hereafter we write (z, ξ, w, η) for coordinates $(z, w; \xi, \eta)$.

2. SYMBOLS OF PRODUCT TYPE

Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. $P(z, \xi, w, \eta)$ is said to be a symbol of product type on K if the following hold:

- (1) There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.
- (2) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$(2.1) \quad |P(z, \xi, w, \eta)| \leq C_\varepsilon e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U; d, d).$$

We denote by $S(K)$ the set of all such symbols on K .
 $S(K)$ becomes a commutative ring with the usual sum and product.

Definition 2.2. We denote by $R(K)$ the set of all $P(z, \xi, w, \eta) \in S(K)$ satisfying the following;
 there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U; d, d)$.

We call an element of $R(K)$ a symbol of 0-class.

Definition 2.3. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on K if the following hold:

- (1) There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.
- (2) For each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$(2.2) \quad |P_{j,k}(z, \xi, w, \eta)| \leq C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U; (j+1)d, (k+1)d)$$

for each $j, k \geq 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K .

We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$, in indeterminants t_1 and t_2 for $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$.

We can easily obtain the following.

Proposition 2.4. $\widehat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in t_1 and t_2 .

$S(K)$ is identified with a subring of $\widehat{S}(K)$ as follows:
 $S(K) \simeq \widehat{S}(K)|_{\substack{t_1=0 \\ t_2=0}} = \{P = \sum t_1^j t_2^k P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j, k) \neq (0, 0)\}.$

Definition 2.5. We denote by $\widehat{R}(K)$ the set of all $P(t_1, t_2; z, \xi, w, \eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$ in $\widehat{S}(K)$ such that there are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left| \sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class.

Proposition 2.6. *Under the previous identification, $S(K) \cap \widehat{R}(K) = R(K)$ holds.*

Proof. Let $P(z, \xi, w, \eta)$ be in $S(K)$. Then $P(z, \xi, w, \eta) \in \widehat{R}(K)$ is equivalent to the following;

there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{|\xi|, |\eta|\} + \varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U; d, d)$.

(\subset) Using the fact that $(0, \infty) = \{t := \frac{|\xi|}{|\eta|}; (z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)\}$, by the hypothesis, we obtain the following;

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{1, \frac{1}{t}\}|\xi| + \varepsilon(1+\frac{1}{t})|\xi|}$$

for all $t := \frac{|\xi|}{|\eta|} \in (0, \infty)$ and $(z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)$.

We fix any $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $\varepsilon \leq \frac{\delta}{3}$.

Then for every $t \in [\frac{3}{\delta}\varepsilon, 1]$

$$\begin{aligned} |P(z, \xi, w, \eta)| &\leq C_\varepsilon e^{-\delta|\xi| + \varepsilon(1+\frac{1}{t})|\xi|} \leq C_\varepsilon e^{-\delta|\xi| + \varepsilon(1+\frac{\delta}{3\varepsilon})|\xi|} \\ &= C_\varepsilon e^{(\varepsilon - \frac{2}{3}\delta)|\xi|} \leq C_\varepsilon e^{-\frac{1}{3}\delta|\xi|}. \end{aligned}$$

On the other hand, for any sequence ε_n such that $\min\{1, \frac{\delta}{3}\} > \varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$, we define a function $C(\cdot)$ on $(0, 1]$ as

$$C(t) := \begin{cases} C_{\frac{3}{\delta}\varepsilon_1}, & \varepsilon_1 < t \leq 1, \\ C_{\frac{3}{\delta}\varepsilon_{n+1}}, & \varepsilon_{n+1} < t \leq \varepsilon_n. \end{cases}$$

Then $C(\cdot)$ is locally bounded on $(0, 1]$ and

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right) e^{-\frac{1}{3}\delta|\xi|}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.

In like manners,

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\eta|}{|\xi|}\right)e^{-\frac{1}{3}\delta|\eta|}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \geq |\eta|\}$.

Here, we define a function $C''(\cdot)$ on $(0, \infty)$ as $C''(t) = C(\min\{t, \frac{1}{t}\})$.

Then $C''(t)$ is locally bounded on $(0, \infty)$

and $|P(z, \xi, w, \eta)| \leq C''\left(\frac{|\xi|}{|\eta|}\right)e^{-\frac{1}{3}\delta \cdot \min\{|\xi|, |\eta|\}}$ on $\gamma^{-1}(U; d, d)$.

That is, $P(z, \xi, w, \eta) \in R(K)$.

(\supset) Let $P(z, \xi, w, \eta) \in R(K)$.

Then there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\delta \min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U; d, d)$. We fix any ε such that $0 < \varepsilon < 1$. Then,

$$|P(z, \xi, w, \eta)| \leq \max_{\varepsilon \leq t \leq 1} C(t) \cdot e^{-\delta \min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U; d, d) \cap \{\varepsilon \leq \frac{|\xi|}{|\eta|} =: t \leq 1\}$. We put $C'_\varepsilon := \max_{\varepsilon \leq t \leq 1} C(t)$.

On the other hand, since $P(z, \xi, w, \eta) \in S(K)$, there exists $C''_\varepsilon > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C''_\varepsilon e^{\varepsilon(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d)$.

Therefore, the following inequalities hold on $\gamma^{-1}(U; d, d) \cap \{\frac{|\xi|}{|\eta|} \leq \varepsilon\}$.

$$\begin{aligned} |P(z, \xi, w, \eta)| &\leq C''_\varepsilon e^{-\delta \min\{(|\xi|, |\eta|)\} + \delta \min\{(|\xi|, |\eta|)\} + \varepsilon(|\xi| + |\eta|)} \\ &\leq C''_\varepsilon e^{-\delta \min\{(|\xi|, |\eta|)\} + \varepsilon(1 + \delta)(|\xi| + |\eta|)}. \end{aligned}$$

If we put $C_\varepsilon := \max\{C'_\varepsilon, C''_\varepsilon\}$,

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{(|\xi|, |\eta|)\} + \varepsilon(1 + \delta)(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.

That is, $P(z, \xi, w, \eta) \in \widehat{R}(K)$.

Proposition 2.7. $R(K)$ is an ideal in $S(K)$.

Proof. It is clear by the part (\subset) of the proof of Proposition 2.6.

Proposition 2.8. $\widehat{R}(K)$ is an ideal in $\widehat{S}(K)$.

Proof. Let $\sum P_{j,k}(z, \xi, w, \eta) \in \widehat{R}(K)$ and $\sum Q_{j,k}(z, \xi, w, \eta) \in \widehat{S}(K)$. Then there exist $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

For each $\varepsilon > 0$, we have some $C_\varepsilon > 0$ such that

$$\text{a) } |P_{s,t}(z, \xi, w, \eta)|, |Q_{s,t}(z, \xi, w, \eta)| \leq C_\varepsilon A^{s+t} e^{\varepsilon(|\xi|+|\eta|)}$$

$$\text{b) } \left| \sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

It suffices to show that $\sum R_{j,k} \in \widehat{R}(K)$, where

$$R_{j,k} := \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2}.$$

Since we can easily estimate $\sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} R_{j,k}$ for $st = 0$,

we suppose $s \geq 1$ and $t \geq 1$.

Then we can obtain the following inequality:

$$\begin{aligned} \left| \sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} R_{j,k} \right| &= \left| \sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right| \\ &\leq \left| \left(\sum_{\substack{0 \leq j_1 \leq s \\ 0 \leq k_1 \leq t}} P_{j_1,k_1} \right) \left(\sum_{\substack{0 \leq j_2 \leq s \\ 0 \leq k_2 \leq t}} Q_{j_2,k_2} \right) \right| + \left| \sum_{\substack{s+1 \leq j \leq 2s \\ t+1 \leq k \leq 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right| \\ &+ \left| \sum_{\substack{0 \leq j \leq s \\ t+1 \leq k \leq 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right| + \left| \sum_{\substack{s+1 \leq j \leq 2s \\ 0 \leq k \leq t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right|. \end{aligned}$$

We shall estimate the four terms in the right side of the inequality, respectively.

$$\begin{aligned} \text{the first term} &\leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)} \cdot \sum_{\substack{0 \leq j_2 \leq s \\ 0 \leq k_2 \leq t}} C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \\ &\leq C_\varepsilon \cdot C_\varepsilon \cdot A^{\min\{s,t\}} e^{2\varepsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A} \end{aligned}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the 2nd term

$$\begin{aligned} &\leq \sum_{\substack{s+1 \leq j \leq 2s \\ t+1 \leq k \leq 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} C_\varepsilon A^{j_1+k_1} e^{\varepsilon(|\xi|+|\eta|)} \cdot C_\varepsilon A^{j_2+k_2} e^{\varepsilon(|\xi|+|\eta|)} \\ &= C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot \left(\sum_{s+1 \leq j \leq 2s} \sum_{j_1+j_2=j} A^j \right) \left(\sum_{t+1 \leq k \leq 2t} \sum_{k_1+k_2=k} A^k \right). \end{aligned}$$

If we choose any B and C such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$\sum_{s+1 \leq j \leq 2s} \sum_{j_1+j_2=j} A^j \leq C^{s+1} (B^0 + B^1 + B^2 + \dots)^2 = C^{s+1} \left(\frac{1}{1-B} \right)^2.$$

Then,

$$\text{the second term} \leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot C^{s+1} \left(\frac{1}{1-B} \right)^2 \cdot C^{t+1} \left(\frac{1}{1-B} \right)^2$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the third term

$$\begin{aligned} &\leq \sum_{\substack{0 \leq j \leq s \\ t+1 \leq k \leq 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} C_\varepsilon A^{j_1+k_1} e^{\varepsilon(|\xi|+|\eta|)} \cdot C_\varepsilon A^{j_2+k_2} e^{\varepsilon(|\xi|+|\eta|)} \\ &= C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \left(\sum_{0 \leq j \leq s} \sum_{j_1+j_2=j} A^j \right) \left(\sum_{t+1 \leq k \leq 2t} \sum_{k_1+k_2=k} A^k \right) \\ &\leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot \left(\frac{1}{1-A} \right)^2 \cdot C^{t+1} \left(\frac{1}{1-B} \right)^2 \end{aligned}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

In like manners,

$$\text{the fourth term} \leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot C^{s+1} \cdot \left(\frac{1}{1-B} \right)^2 \cdot \left(\frac{1}{1-A} \right)^2$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

Hence, we conclude that $\sum R_{j,k} \in \widehat{R}(k)$.

$\widehat{S}(K)/\widehat{R}(K)$ becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion $S(K) \hookrightarrow \widehat{S}(K)$ induces the injective ring homomorphism

$$\iota_K : S(K)/R(K) \longrightarrow \widehat{S}(K)/\widehat{R}(K).$$

Conversely, we obtain the following.

Theorem 2.9. *If $\sum P_{j,k}(z, \xi, w, \eta) \in \widehat{S}(K)$, there exists $P(z, \xi, w, \eta) \in S(K)$ such that $P - \sum P_{j,k} \in \widehat{R}(K)$.*

Thus, $S(K)/R(K)$ is isomorphic to $\widehat{S}(K)/\widehat{R}(K)$ in the sense of commutative rings.

Definition 2.10. We call an element in the ring $\widehat{S}(K)/\widehat{R}(K)$ a pseudo-differential operator of the product type on K . We write $:\sum P_{j,k}:$ for the associated pseudo-differential operator of the product type on K using an element $\sum P_{j,k}$ in $\widehat{S}(K)$.

The mapping γ is the composition of the following γ_1 and γ_2 .

$$\overset{\circ}{T}^*(X \times Y) \ni (z, w; \xi, \eta) \xrightarrow{\gamma_2} (z, w; \frac{\xi}{|(\xi, \eta)|}, \frac{\eta}{|(\xi, \eta)|}) \in \overset{\circ}{S}^*(X \times Y),$$

$$\overset{\circ}{S}^*(X \times Y) \ni (z, w; \frac{\xi}{|(\xi, \eta)|}, \frac{\eta}{|(\xi, \eta)|}) \xrightarrow{\gamma_1} (z, \frac{\xi}{|\xi|}) \times (w, \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

$$\text{where } \overset{\circ}{S}^*(X \times Y) := S^*(X \times Y) \setminus \{(S^*X \times Y) \cup (X \times S^*Y)\}.$$

Proposition 2.11. *If $P(z, \xi, w, \eta)$ is a symbol of product type on K , P is a symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.*

Proof. By the hypothesis, there are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

- a) $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and
- b) for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{\varepsilon(|\xi|+|\eta|)}$ on $\gamma^{-1}(U; d, d)$.

Let K' be compact in $\overset{\circ}{S}^*(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$.

Then there exist $d' > 0$ and $U' \supset K'$ open in $\overset{\circ}{S}^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.$$

In fact, for each $(\overset{\circ}{z}, \overset{\circ}{w}; \overset{\circ}{\xi}, \overset{\circ}{\eta}) \in \gamma_1^{-1}(K)$ we can choose $d' > 0$ such that

$$d' > \frac{d}{\min\{|\overset{\circ}{\xi}|, |\overset{\circ}{\eta}|\}}.$$

Then there exists a neighborhood U' of $(\overset{\circ}{z}, \overset{\circ}{w}; \overset{\circ}{\xi}, \overset{\circ}{\eta}) \in \gamma_1^{-1}(K)$ in $\overset{\circ\circ}{S}^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.$$

By the compactness of K' , the proof is completed.

Proposition 2.12. *If $P(z, \xi, w, \eta)$ is a symbol of product type of 0-class on K , that is, $P \in R(K)$, P is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.*

Proof. Let K' be compact in $\overset{\circ\circ}{S}^*(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$.

It suffices to show that P is a zero symbol on K' in the sense of AOKI's symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right) e^{-\delta \min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$.

Let $(\overset{\circ}{z}, \overset{\circ}{w}; \overset{\circ}{\xi}, \overset{\circ}{\eta})$ be any point of $\gamma_1^{-1}(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood U' of $(\overset{\circ}{z}, \overset{\circ}{w}; \overset{\circ}{\xi}, \overset{\circ}{\eta})$ in $\overset{\circ\circ}{S}^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\},$$

and that there exists $\delta' > 0$ satisfying

$$\min\left\{\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|}\right\} > \delta' \quad \text{on} \quad \gamma_2^{-1}(U').$$

Hence,

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right) e^{-\delta\delta'(|\xi| + |\eta|)}$$

on $\gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}$.

Since K is compact, P is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

Definition 2.13. The canonical mapping H_K is defined as follows;

$$S(K)/R(K) \ni P \xrightarrow{H_K} [P] \in \varinjlim_{U \supset \gamma_1^{-1}(K)} \mathcal{E}^{\mathbb{R}}(U).$$

Proposition 2.14. *Suppose K_1 and K_2 are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(: P :) \Big|_{\gamma_1^{-1}(K_2)} = H_{K_2}(: P|_{K_2} :)$ for all $P \in S(K)/R(K)$.*

Definition 2.15. We define the product $*$ of two elements of $\widehat{S}(K)$. as follows:

$$\left(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)\right) * \left(\sum_{j,k=0}^{\infty} Q_{j,k}(z, \xi, w, \eta)\right) = \sum_{j,k=0}^{\infty} R_{j,k}(z, \xi, w, \eta),$$

where

$$\begin{aligned} \sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z, \xi, w, \eta) &:= e^{t_1 \langle \partial_{\xi}, \partial_{z^*} \rangle + t_2 \langle \partial_{\eta}, \partial_{w^*} \rangle} \left(\left(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta) \right) \right. \\ &\quad \times \left. \left(\sum_{j,k=0}^{\infty} Q_{j,k}(z^*, \xi^*, w^*, \eta^*) \right) \right) \Bigg|_{\substack{z^*=z, \xi^*=\xi \\ w^*=w, \eta^*=\eta}}. \end{aligned}$$

That is,

$$\begin{aligned} R_{j,k}(z, \xi, w, \eta) &:= \sum_{\substack{j_1+j_2+|\alpha|=j \\ k_1+k_2+|\beta|=k}} \frac{1}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} P_{j_1 k_1}(z, \xi, w, \eta) \\ &\quad \times \partial_z^{\alpha} \partial_w^{\beta} Q_{j_2 k_2}(z, \xi, w, \eta). \end{aligned}$$

Then we obtain the following.

Lemma 2.16. If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on K , then $\sum R_{j,k}$ is also a formal symbol of product type on K .

Proposition 2.17. If $\sum P_{j,k} \in \widehat{S}(K)$ and $\sum Q_{j,k} \in \widehat{R}(K)$, otherwise $\sum P_{j,k} \in \widehat{R}(K)$ and $\sum Q_{j,k} \in \widehat{S}(K)$, $\sum R_{j,k}$ is also in $\widehat{R}(K)$.

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in $\widehat{S}(K)/\widehat{R}(K)$ is well-defined;

$$: \sum P_{j,k} : \circ : \sum Q_{j,k} ::= (\sum P_{j,k}) * (\sum Q_{j,k}) : .$$

We can easily verify the associativity about the operation \circ . That is, $\widehat{S}(K)/\widehat{R}(K)$ becomes an associative \mathbb{C} algebra. Hence the mapping H_K is a homomorphism about the operation \circ , $+$, and \cdot , where

$$\mathcal{E}_{X \times Y}^{\mathbb{R}, prod}(K) \equiv \widehat{S}(K)/\widehat{R}(K) \xrightarrow{H_K} \mathcal{E}_{X \times Y}^{\mathbb{R}}(\gamma^{-1}(K)).$$

Definition 2.18. The reverse of $\sum P_{j,k}$ in $\widehat{S}(K)$ is defined as

$$\left(\sum t_1^j t_2^k P_{j,k}\right)^R := e^{t_1 \langle \partial_{\xi}, \partial_z \rangle + t_2 \langle \partial_{\eta}, \partial_w \rangle} \left(\sum t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)\right).$$

We can verify that if $\sum P_{j,k}$ is in $\widehat{S}(K)$ ($\widehat{R}(K)$) then $(\sum P_{j,k})^R$ is in $\widehat{S}(K)$ ($\widehat{R}(K)$), respectively.

3. EXPONENTIAL CALCULUS OF SYMBOLS OF MINIMUM TYPE

Definition 3.1. A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold;

- (1) Λ is continuous,
- (2) for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,
- (3) Λ is increasing,
- (4) $\lim_{t \rightarrow \infty} \frac{\Lambda(t)}{t} = 0$.

Definition 3.2. $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order (Λ_1, Λ_2) on K if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

- (1) $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and
- (2) $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

Example 3.3. (by K. Kataoka)

$$\Omega = \Omega' := \mathbb{C} \times \{\xi \in \mathbb{C}; |\arg \xi| < \delta, \xi \neq 0\} (0 < \delta < \frac{\pi}{2}).$$

Let K be any compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.

$$P(z, \xi, w, \eta) := (\xi\eta)^{(1+\sigma)/2}/(\xi + \eta),$$

$$\Lambda_1(t) = \Lambda_2(t) := t^\sigma \text{ with } 0 < \sigma < 1.$$

Remark 3.4. If P is a symbol of minimum type on K , e^P is a symbol of product type on K .

Definition 3.5. $\sum P_{j,k}$ in $\widehat{S}(K)$ is called a formal symbol of minimum type of growth order (Λ_1, Λ_2) on K if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

- (1) $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
- (2)

$$|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$$

on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

Remark 3.6. If $\sum P_{j,k}$ is a formal symbol of minimum type on K , $e^{\sum P_{j,k}}$ is a formal symbol of product type on K .

Proposition 3.7. *If P and Q are in $S(K)$, then*

$$(3.1) \quad \begin{aligned} & P(z, \xi, w, \eta) * (Q(z, \xi, w, \eta))^R \\ &= e^{t_1 \langle \partial_\xi, \partial_{z^*} \rangle + t_2 \langle \partial_\eta, \partial_{w^*} \rangle} P(z, \xi, w, \eta) Q(z^*, \xi, w^*, \eta) \Big|_{\substack{z^*=z \\ w^*=w}}. \end{aligned}$$

Theorem 3.8. *If P and Q are symbols of minimum type of growth order (Λ_1, Λ_2) on K , there exists a formal symbol, $\sum R_{j,k}$, of minimum type on K satisfying $e^P * e^Q = e^{\sum t_1^j t_2^k R_{j,k}}$.*

Proof. $W(s, t; z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*)$
 $:= e^{s \langle \partial_\xi, \partial_{z^*} \rangle + t \langle \partial_\eta, \partial_{w^*} \rangle} \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*))$ is the unique formal series solution to the following system of partial differential equations:

$$(3.2) \quad \begin{cases} \partial_s W = \langle \partial_\xi, \partial_{z^*} \rangle W, & \partial_t W = \langle \partial_\eta, \partial_{w^*} \rangle W, \\ W_{s=t=0} = \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*)). \end{cases}$$

If we put $W = \exp(\sum_{j,k} s^j t^k W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k \geq 0}$:

$$(3.3) \quad \begin{cases} W_{0,0} = P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*), \\ W_{j+1,k} = \frac{1}{j+1} \{ \langle \partial_\xi, \partial_{z^*} \rangle W_{j,k} + \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} \langle \partial_\xi W_{j_1,k_1}, \partial_{z^*} W_{j_2,k_2} \rangle \}, \\ W_{j,k+1} = \frac{1}{k+1} \{ \langle \partial_\eta, \partial_{w^*} \rangle W_{j,k} + \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} \langle \partial_\eta W_{j_1,k_1}, \partial_{w^*} W_{j_2,k_2} \rangle \}. \end{cases}$$

Then $R_{j,k} = W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) \Big|_{\substack{z^*=z, \xi^*=\xi \\ w^*=w, \eta^*=\eta}}$.

Suppose there exist $C_P (= C_Q) > 0$, $d > 0$, and an open subset $U (\supset K)$ of $S^*X \times S^*Y$ satisfying the following:

- (1) P and Q are holomorphic in $\gamma^{-1}(U; d, d)$,
- (2) $|P(z, \xi, w, \eta)|$, and $|Q(z, \xi, w, \eta)| \leq C_P \tilde{\Lambda}(|\xi|, |\eta|)$ on $\gamma^{-1}(U; d, d)$,
 where $\tilde{\Lambda}(|\xi|, |\eta|) := \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$.

$V := \gamma^{-1}(U; d, d) \times \gamma^{-1}(U; d, d)$,
 $V^{\varepsilon_1, \varepsilon_2} := \{(z, w, \xi, \eta, z^*, w^*, \xi^*, \eta^*) \in V; |\xi' - \xi| \leq \varepsilon_1 |\xi|, |z^{*'} - z^*| \leq \varepsilon_1, |\eta' - \eta| \leq \varepsilon_2 |\eta|, |w^{*'} - w^*| \leq \varepsilon_2 \implies (z, w, \xi', \eta', z^{*'}, w^{*'}, \xi^*, \eta^*) \in V\}$.

Then we obtain the following lemma.

Lemma 3.9. Suppose $\{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0}$ satisfy the following conditions:

(1)

$$C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1} \{C_{j,k}^{(\mu,\nu)}(j+1)^2 + \sum^* (j_1+1)(j_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)}\},$$

(2)

$$C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1} \{C_{j,k}^{(\mu,\nu)}(k+1)^2 + \sum^{**} (k_1+1)(k_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)}\},$$

$$(3) \quad C_{0,0}^{0,0} \leq C_P + C_Q,$$

$$(4) \quad C_{j,k}^{(\mu,\nu)} \geq 0 \quad (j, k \geq 0, \quad 0 \leq \mu \leq j, \quad 0 \leq \nu \leq k),$$

$$(5) \quad C_{j,k}^{(\mu,\nu)} = 0 \quad (\text{otherwise}).$$

Here, the sum \sum^* , \sum^{**} mean $\sum_{\substack{j_1+j_2=j, \\ k_1+k_2=k, \\ \mu_1+\mu_2=\mu-1, \\ \nu_1+\nu_2=\nu}}$, $\sum_{\substack{j_1+j_2=j, \\ k_1+k_2=k, \\ \mu_1+\mu_2=\mu, \\ \nu_1+\nu_2=\nu-1}}$, respectively.

Then for each ε_1 and ε_2 such that $0 < \varepsilon_1 \ll 1$ and $0 < \varepsilon_2 \ll 1$, the following hold:

$$(3.4) \quad |W_{j,k}| \leq \sum_{\substack{0 \leq \mu \leq j, \\ 0 \leq \nu \leq k}} \frac{C_{j,k}^{(\mu,\nu)}}{\varepsilon_1^{2j} \varepsilon_2^{2k} |\xi|^j |\eta|^k} (\tilde{\Lambda}(|\xi|, |\eta|) + \tilde{\Lambda}(|\xi^*|, |\eta^*|)) \\ \times (\Lambda_1(|\xi|) + \Lambda_1(|\xi^*|))^\mu (\Lambda_2(|\eta|) + \Lambda_2(|\eta^*|))^\nu$$

on $V^{\varepsilon_1, \varepsilon_2}$ for all $j, k \geq 0$.

The following lemma guarantees the existence of $C_{j,k}^{(\mu,\nu)}$ satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. The following sequence $\{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0}$ satisfies the conditions from (1) through (5) of the previous lemma.

$$C_{j,k}^{(\mu,\nu)} := \begin{cases} lB^{j+k}(j+1)^{j-\mu-3}(k+1)^{k-\nu-3}, & (0 \leq \mu \leq j, \quad 0 \leq \nu \leq k) \\ 0, & (\text{otherwise}) \end{cases},$$

where l and B are constants satisfying $l \geq \max\{C_P + C_Q, 1\}$ and $B \geq 72l \cdot \max\{m, n\} \cdot e^{10} \cdot (c^2 + 1)$ and c is a constant satisfying the following T.Aoki's inequality;

$$(3.5) \quad \frac{1}{j+1} \sum_{\mu=0}^{\nu-1} \sum_{k=\mu}^{j-\nu+\mu+1} (k+1)^{k-\mu-2} (j-k+1)^{j-k-\nu+\mu-1} \leq c(j+1)^{j-\nu-2}$$

for all j, ν such that $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

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